(3) The function $f(z) = \dfrac{(\log z)^3}{z^2+1}$ has isolated singularities at $z = i, -i$. Here $\log z$ is the branch

$$\log z = \ln r + i\theta \quad (r>0, \ 0<\theta<2\pi).$$

Consider $z=i$. Define $\phi(z) = \dfrac{(\log z)^3}{z+i}$. Then

$$f(z) = \frac{\phi(z)}{z-i}.$$

Clearly ϕ is analytic at $z=i$ and $\phi(i) \neq 0$ since

$$\phi(i) = \frac{(\log i)^3}{2i} = \frac{(\ln|i| + i\pi/2)^3}{2i}$$

$$= \frac{i^3 \cdot \frac{\pi^3}{8}}{2i} = -\frac{\pi^3}{16} \neq 0.$$

By the theorem, $z=i$ is a simple pole. The residue is given by

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{-\pi^3}{16}.$$

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Zeros of Analytic Functions

Definition (Zeros of Analytic Functions) Assume f is analytic at $z_0 \in \mathbb{C}$. We say that f has a zero of order m if $f(z_0)=0$ and there exist $m \geq 1$ such that $f^{(m)}(z_0) \neq 0$ but $f^{(n)}(z_0) = 0$ for all $0 \leq n < m$. A zero is isolated if there exists $\varepsilon > 0$ such that $f(z) \neq 0$ for all $z \in D_\varepsilon(z_0) \setminus \{z_0\}$.

Theorem (Characterization of Zeros) Suppose that f is analytic at z_0. The following are equivalent:

(a) z_0 is a zero of f of order m.

(b) $f(z) = (z-z_0)^m g(z)$ for some function $g(z)$ analytic and nonzero at z_0.

Proof. ($a \Rightarrow b$) Assume z_0 is a zero of order m. Since f is analytic at z_0, f has a Taylor series on some disk $D_\varepsilon(z_0)$:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \\ &= \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \\ &= (z-z_0)^m \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m}. \end{aligned}$$

Define $g(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$. Clearly $g(z)$ is analytic at z_0 since it converges on $D_\varepsilon(z_0)$. Moreover, $g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$ Since z_0 is a pole of order m.

($b \Rightarrow a$) Assume $f(z) = (z-z_0)^m g(z)$ where g is analytic and nonzero at z_0. Since g is analytic at z_0, there is a disk $D_\varepsilon(z_0)$ on which it has a Taylor series. Then

$$\begin{aligned} f(z) &= (z-z_0)^m g(z) \\ &= (z-z_0)^m \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^{n+m}. \end{aligned}$$

Since Taylor series are unique, the coefficients in this power series for f are the ones given by Taylors theorem. Hence,

$$\frac{f^{(n)}(z_0)}{n!} = 0 \quad \text{for all} \quad n = 0, 1, \dots, m-1$$

and $\dfrac{f^{(m)}(z_0)}{m!} = g(z_0)$. Hence, $f^{(m)}(z_0) = g(z_0) \cdot m! \neq 0$

and $f^{(n)}(z_0) = 0$ for all $0 \le n < m-1$. ■

$\boxed{\text{Example}}$ The function $p(z) = z^3 - 1$ has a zero of order $m = 1$ at $z_0 = 1$. Just define $g(z) = z^2 + z + 1$. Then

$$p(z) = (z-1)(z^2 + z + 1) = (z-1)\,g(z).$$

Clearly $g(z)$ is analytic at $z_0 = 1$ and

$$g(1) = 3 \neq 0.$$

So by the theorem $p(z)$ has a zero of order 1 at $z_0 = 1$.

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$\boxed{\text{Theorem (Zeros of nonzero Analytic Functions)}}$ Suppose that

(a) f is analytic at z_0;

(b) $f(z_0) = 0$, but f is not identically zero on any neighborhood of z_0.

Then z_0 is an isolated zero of f.

Proof. By (a) there is a disk $|z - z_0| < R$ on which we can write

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

If $f^{(n)}(z_0) = 0$ for all $n \geq 0$, then $f(z)$ would be identically zero on $|z-z_0| < R$, contrary to (b). Hence, there exists $m \geq 1$ such that

$$f(z_0) = f^{(1)}(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \quad \text{but}$$

$f^{(m)}(z_0) \neq 0$. Hence f has a zero of order m. Then

$$f(z) = (z-z_0)^m g(z)$$

for some function $g(z)$ that is analytic and nonzero at z_0. Since g is continuous and nonzero at z_0, there exists a disk $D_\varepsilon(z_0)$ on which $g(z) \neq 0$ for all $z \in D_\varepsilon(z_0)$. Hence, $f(z) \neq 0$ on the deleted disk $D_\varepsilon(z_0) \setminus \{z_0\}$. Hence, z_0 is an isolated zero of f.

∎

Zeros and Poles

$\boxed{\text{Theorem (Zeros and Poles)}}$ Suppose that

(a) $p(z)$ and $q(z)$ are analytic at $z_0 \in \mathbb{C}$.

(b) $p(z_0) \neq 0$ and $q(z)$ has a zero of order m at z_0.

Then $f(z) = \dfrac{p(z)}{q(z)}$ has a pole of order m at z_0.

Proof. First, since $q(z)$ is analytic at z_0 and has a zero of order m at z_0, by the preceding theorem z_0 is an isolated zero. Hence f has an isolated singularity at z_0. Since z_0 is a zero of order m, choose $g(z)$ that is analytic and nonzero at z_0 such that

$$q(z) = (z-z_0)^m g(z).$$

Hence, write $\phi(z) = \dfrac{p(z)}{g(z)}$ so that we have

$$f(z) = \frac{p(z)}{g(z)} = \frac{p(z)/g(z)}{(z-z_0)^m} = \frac{\phi(z)}{(z-z_0)^m}.$$

Moreover, $\phi(z_0) \neq 0$ and it is analytic at z_0 since both $p(z)$ and $g(z)$ are. Hence f has a pole of order m. ∎

$\boxed{\text{Example}}$ Consider $f(z) = \frac{1}{1-\cos z}$. Using the theorem, we can show that f has a pole of order $m=2$ at $z_0 = 0$.

Let $p(z)=1$ and $g(z) = 1-\cos z$. Clearly, both $p(z)$ and $g(z)$ are analytic at $z_0=0$. Moreover,

$$p(0) = 1 \neq 0$$

and $g(z)$ has a zero of order $m=2$ since

$$g(0) = 1 - \cos 0 = 0$$
$$g'(0) = \sin 0 = 0$$
$$g''(0) = \cos 0 = 1 \neq 0,$$

By the theorem, $f(z)$ has a pole of order $m=2$ at $z_0=0$.

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$\boxed{\text{Theorem (Residue at a Simple Pole)}}$ Suppose $p(z), g(z)$ are analytic at z_0. If

$$p(z_0) \neq 0, \quad g(z_0) = 0, \quad \text{and} \quad g'(z_0) \neq 0,$$

then z_0 is a simple pole of $\frac{p(z)}{g(z)}$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{g(z)} = \frac{p(z_0)}{g'(z_0)}.$$

Proof. First, $g(z)$ has a zero of order $m=1$ since $g(z_0)=0$

and $g'(z_0) \neq 0$. Choose a function $g(z)$ that is analytic and nonzero at z_0 such that

$$q(z) = (z-z_0)\, g(z). \quad (*)$$

Using $\phi(z) = \frac{p(z)}{g(z)}$ we can conclude that

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{z-z_0}$$

(see proof of preceding thm)

and that $\frac{p(z)}{q(z)}$ has a pole of order $m=1$. Hence,

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \phi(z_0) = \frac{p(z_0)}{g(z_0)}.$$

But from $(*)$, we obtain

$$q'(z_0) = q'(z)\Big|_{z=z_0} = \frac{d}{dz}(z-z_0)\,g(z)\Big|_{z=z_0}$$

$$= \Big[g(z) + g'(z)(z-z_0)\Big]\Big|_{z=z_0}$$

$$= g(z_0).$$

Hence,

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

■

Example

(1) Consider $f(z) = \cot z = \frac{\cos z}{\sin z}$. Let $p(z) = \cos z$ and $q(z) = \sin z$. Let $z_k = k\pi$, $k \in \mathbb{Z}$. Clearly, both p and q are analytic at z_k, because they are entire. Moreover,

$$p(z_k) = \cos k\pi = (-1)^k \neq 0$$

$$q(z_k) = \sin k\pi = 0$$

$$q'(z_k) = \cos k\pi = (-1)^k \neq 0.$$

Hence, z_k is a simple pole for each $k \in \mathbb{Z}$ and

$$\operatorname*{Res}_{z=z_k} \cot z = \frac{p(z_k)}{q'(z_k)} = \frac{(-1)^k}{(-1)^k} = 1.$$

Let C be the positively oriented circle of radius $k\pi + 1$ centered at 0. Then

$$\int_C \cot z \, dz = 2\pi i \sum_{n=-k}^{k} \operatorname*{Res}_{z=z_k} \cot z$$
$$= 2\pi i \, (2k+1).$$

(2) Consider $f(z) = \dfrac{z - \sinh z}{z^2 \sinh z}$. Consider $z = \pi i$ and

let $p(z) = z - \sinh z$ and $q(z) = z^2 \sinh z$. Both p and q are entire and hence analytic at $z = \pi i$. Moreover,

$$p(\pi i) = \pi i - \sinh \pi i = \pi i \neq 0$$
$$q(\pi i) = (\pi i)^2 \sinh \pi i = 0$$
$$q'(\pi i) = \left(2z \sinh z + z^2 \cosh z\right)\Big|_{z=\pi i}$$
$$= (\pi i)^2 \cosh \pi i$$
$$= -\pi^2 \left(\frac{e^{\pi i} + e^{-\pi i}}{2} \right)$$
$$= -\pi^2 \left(\frac{-1 - 1}{2} \right) = \pi^2 \neq 0.$$

So $z = \pi i$ is a simple pole and

$$\operatorname*{Res}_{z=\pi i} f(z) = \frac{p(\pi i)}{q'(\pi i)} = \frac{\pi i}{\pi^2} = \frac{i}{\pi}.$$

(3) Consider $f(z) = \dfrac{z}{z^4 + 4}$. Consider $z = 1 + i$. Let

$p(z) = z$ and $q(z) = z^4 + 4$. Then

$$p(1+i) = 1+i \neq 0$$
$$q(1+i) = 0$$
$$q'(1+i) = 4(1+i)^3 \neq 0.$$

So $z = 1+i$ is a simple pole and

$$\operatorname*{Res}_{z=1+i} f(z) = \frac{p(1+i)}{q'(1+i)} = \frac{1}{4(1+i)^2}.$$

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Chapter 7: Applications of Residue Theory

We will now apply the theory of residues to compute several types of improper integrals from real analysis.

Additionally, we will prove:

(1) Argument Principle - The winding number of the image of a curve under certain analytic functions depends only on the number of zeros and poles of that function.

(2) Rouche's Theorem - A useful criterion for locating the zeros of an analytic function.

Background on Improper Integrals

Definition Suppose $f(x)$ is a real-valued function of a real variable.

(a) If $f(x)$ is continuous on $[0,\infty)$, then the improper integral of f over that interval is defined to be

$$\int_0^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{R\to\infty} \int_0^R f(x)\,dx.$$

If the limit exists, the integral is said to converge.

(b) If $f(x)$ is continuous on $\mathbb{R}$, then the improper integral of f over $\mathbb{R}$ is defined via

$$\int_{-\infty}^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)\,dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\,dx$$

The integral converges if both limits exist.

(c) The Cauchy Principal Value of the improper integral in (b) is the value of the limit

$$\text{P.V.} \int_{-\infty}^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{R\to\infty} \int_{-R}^R f(x)\,dx.$$

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Lemma If $\int_{-\infty}^\infty f(x)\,dx$ converges, then the Cauchy principal value exists and $\text{P.V.}\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^\infty f(x)\,dx$.

Proof. Just notice

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx + \lim_{R\to\infty} \int_{-R}^0 f(x)\,dx$$

$$= \lim_{R\to\infty} \left(\int_0^R f(x)\,dx + \int_{-R}^0 f(x)\,dx \right)$$

$$= \lim_{R\to\infty} \int_{-R}^R f(x)\,dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx.$$

$\blacksquare$

The converse is false - even if the Cauchy Principal value exists, the integral may diverge.